A Continuous Version of a Result of Du and Hwang

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Abstract. In the proof of Gilbert–Pollak conjecture on the Steiner ratio, a result of Du and Hwang on a minimax problem played an important role. In this note, we prove a continuous version of this result.

The Steiner tree problem is a classic intractable problem [6] with many applications in the design of computer circuits, long-distance telephone lines, or mail routing, etc. Given a set of points in a metric space, the problem is to find a shortest network interconnecting the points in the set. Such a shortest network is called the *Steiner minimum tree* (SMT) on the point set.

A *minimum spanning tree* on a set of points is the shortest network interconnecting the given points with all edges between the points. While the Steiner tree problem is intractable, the minimum spanning tree can be computed pretty fast. The Steiner ratio in a metric space is the largest lower bound for the ratio between lengths of a minimum Steiner tree and a minimum spanning tree for the same set of points in the metric space, which is a measure of performance for the minimum spanning tree as a polynomial-time approximation of the minimum Steiner tree. Determining the Steiner ratio in the Euclidean plane was a hard problem. Gilbert and Pollak [7] conjectured that it equals $\sqrt{\frac{3}{2}}$. Through many efforts [1, 2, 3, 5, 8, 9, 10], Du and Hwang [4] finally proved this conjecture.

In Du and Hwang's proof, a new approach was discovered [4]. The center part of this approach is a new theorem about the following minimax problem:

$$\min_{x \in X} \max_{i \in I} f_i(x)$$

where X is a convex region in the n-dimensional Euclidean space R^n, I is a finite index set, and the $f_i(x)$'s are continuous functions over X. The theorem can be stated as follows.

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THEOREM 1 (Du and Hwang). *Let* $g(x) = \max_{i \in I} f_i(x)$. *If every* $f_i(x)$ *is a concave function, then the minimum value of* $g(x)$ *over the polytope* X *is achieved at some point* x^* *satisfying the following condition:*

(*) *There exists an extreme subset* Z *of* X *such that* $x^* \in Z$ *and the set* $I(x^*)(=\{i \mid g(x^*) = f_i(x^*)\})$ *is maximal over* Z.

Here, a subset Z of X is called an *extreme subset* of X if

$$\left.\begin{array}{l} x, y \in X \\ \lambda x + (1-\lambda)y \in Z \text{ for some } 0 < \lambda < 1 \end{array}\right\} \Rightarrow x, y \in Y .$$

In this note, we prove the following continuous version.

THEOREM 2. *Let* $f(x, y)$ *be a continuous function on* $X \times Y$ *where* X *is a polytope in* R^m *and* Y *is a compact set in* R^n. *Let* $g(x) = \max_{y \in Y} f(x, y)$. *If* $f(x, y)$ *is concave with respect to* x, *then the minimum value of* $g(x)$ *over* X *is achieved at some point* $\hat{x}$ *satisfying the following condition*:

(*) *There exists an extreme subset* Z *of* X *such that* $\hat{x} \in Z$ *and the set* $I(\hat{x})$ $(=\{y \mid g(\hat{x}) = f(\hat{x}, y)\})$ *is maximal over* Z.

Proof. Suppose that x^* is a minimum point for $g(x)$ on X. Define $A(y) = \{x \in X \mid f(x, y) \geq g(x^*)\}$. Then, each $A(y)$ is a convex set. Let Z be an extreme set of X such that x^* is a *relative interior point* of Z, that is, for any $x \in Z$ and for sufficiently small positive number λ, $x^* + \lambda(x^* - x) \in Z$. We first show that there exists a point $\hat{x}$ in Z satisfying the condition (*) such that $I(x^*) \subseteq I(\hat{x})$. To show it, let us consider the partial ordering $\subseteq$ on $B = \{I(x) \mid I(x^*) \subset I(x), x \in Z\}$. Since the polytope X must be a compact set, so is Z. Thus, for each sequence

$$I(x^*) \subseteq I(x_1) \subseteq I(x_2) \subseteq \cdots$$

there exists a subsequence $\{x_{k_i}\}$ converging to a point $x' \in Z$. It follows that $I(x_k) \subseteq I(x')$ for all k. By Zorn's lemma, B has a maximal element $I(\hat{x})$. Clearly, $\hat{x}$ is a point in Z satisfying the condition (*). Next, we prove the theorem by proving that $\hat{x}$ is also a minimum point.

For contradiction, suppose that $\hat{x}$ is not a minimum point. Then, for every $y \in I(x^*)$, $\hat{x}$ is an interior point of $A(y)$. Denote $x(\lambda) = x^* + \lambda(x^* - \hat{x})$. We claim that for any positive number λ, $x(\lambda)$ is not in $A(y)$ for every $y \in I(x^*)$. In fact, if the point $x(\lambda)$ for some positive λ is in $A(y)$, then the point x^* as an interior point of the segment $[\hat{x}, x(\lambda)]$ can be written as

$$x^* = c\hat{x} + (1-c)x(\lambda)$$

where $0 < c = \lambda/(1+\lambda) < 1$. Thus, we have

$$f(x^*, y) \geq cf(\hat{x}, y) + (1-c)f(x(\lambda), y) > f(x^*, y) ,$$

a contradiction. Since $x(\lambda)$ is in Z for sufficiently small positive number λ, there exists $\lambda \in (0, \lambda')$ such that $x(\lambda) \in Z \backslash \bigcup_{y \in I(x^*)} A(y)$. For such λ, we must have $I(x(\lambda)) \cap I(x^*) = \emptyset$ because otherwise $g(x(\lambda)) < g(x^*)$, contradicting the minimality of $g(x^*)$. For each λ, choose $y(\lambda) \in I(x(\lambda))$. Since Y is compact, there exists a sequence $\{\lambda_k\}$, converging to 0, such that $y(\lambda_k)$ converges to $y^* \in Y$. Note that $g(x(\lambda_k)) = f(x(\lambda_k), y(\lambda_k))$. Letting $k \to \infty$, we obtain $g(x^*) = f(x^*, y^*)$. Thus, $y^* \in I(x^*)$. Since $I(x^*) \subseteq I(\hat{x})$, it follows that $f(\hat{x}, y^*) = g(\hat{x}) > g(x^*)$. Therefore, for sufficiently large k,

$$f(\hat{x}, y(\lambda_k)) > g(x^*) .$$

Moreover, since $y(\lambda_k) \in I(x(\lambda_k))$ and $y(\lambda_k) \not\in I(x^*)$, we have

$$f(x(\lambda_k), y(\lambda_k)) = g(x(\lambda_k)) \geq g(x^*),$$

and

$$f(x^*, y(\lambda_k)) < g(x^*) .$$

Thus,

$$f(x^*, y(\lambda_k)) < \min[f(x(\lambda_k), y(\lambda_k)), f(\hat{x}, y(\lambda_k))] . \tag{1}$$

Since x^* is at the segment $[x(\lambda_k), \hat{x}]$, (1) contradicts the fact that $f(x, y)$ is concave with respect to x. □

References

1. F. R. K. Chung and R. L. Graham (1985), A new bound for euclidean Steiner minimum trees, *Ann. N. Y. Acad. Sci.* **440**, 328–346.
2. F. R. K. Chung and F. K. Hwang (1978), A lower bound for the Steiner tree problem, *SIAM J. Appl. Math.*, **34**, 27–36.
3. D. Z. Du and F. K. Hwang (1983), A new bound for the Steiner ratio, *Trans. Amer. Math. Soc.* **278**, 137–148.
4. D. Z. Du and F. K. Hwang (1990), An approach for proving lower bounds: solution of Gilbert–Pollak's conjecture on Steiner ratio, *Proceedings 31th FOCS*, 76–85.
5. J. Friedel and P. Widmayer (1989), A simple proof of the Steiner ratio conjecture for five points, *SIAM J. Appl. Math.* **49**, 960–967.
6. M. R. Garey, R. L. Graham, and D. S. Johnson (1977), The complexity of computing Steiner minimal trees, *SIAM J. Appl. Math.* **32**, 835–859.
7. E. N. Gilbert and H. O. Pollak (1968), Steiner minimal trees, *SIAM J. Appl. Math.* **16**, 1–29.
8. R. L. Graham and F. K. Hwang (1976), Remarks on Steiner minimal trees, *Bull. Inst. Math. Acad. Sinica* **4**, 177–182.
9. H. O. Pollak (1978), Some remarks on the Steiner problem, *J. Combinatorial Theory, Ser. A*, **24**, 278–295.
10. J. H. Rubinstein and D. A. Thomas (1991), The Steiner ratio conjecture for six points, *J. of Combinatorial Theory, Series A*, **58**, 54–77.